LUCAS QUOTIENT LEMMAS

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In current work pertaining to models of asymmetric cell division and recursive phyllotaxic patterning in biologic structures [3], [4], number sequences containing F_n entries were organized in rectangular tables with F_m columns. Analysis of data arising from the division of one positive Fibonacci number by another gave a surprising relationship to Lucas numbers: quotients round off to Lucas numbers. That the remainders after such divisions are Fibonacci numbers was known from [1], [2], and [5], but the almost-Lucas quotients in the lemmas following seem to be new.

Lucas Quotient Lemma 1. When F_p is divided by F_m, $3m > p \geq m > 0$, the quotient rounds off (either up or down) to a Lucas number. The remainder is a Fibonacci number or its negative.

Proof: Vajda [6] lists the twin equations (15a) and (15b)

$$F_{n+m} + (-1)^m F_{n-m} = L_m F_n \text{ and } F_{n+m} - (-1)^m F_{n-m} = F_m L_n.$$

Upon division by F_m, (15b) gives

$$F_{n+m}/F_m = L_n + (-1)^m \, F_{n-m}/F_m.$$

If $F_{n-m} < F_m$, the quotient is L_n and the remainder is $\pm F_{n-m}$. If $p = n + m$, the equation above becomes

$$F_p/F_m = L_{p-m} + (-1)^m F_{p-2m}/F_m$$

and when $p < 3m$ so that $p - 2m < m$, the fractional part is less than one in absolute value, and the expression rounds off (either up or down) to L_{p-m}. Note that, when m is even, the remainder is the Fibonacci number F_{p-2m} and we round down; if $F_{p-2m} < 0$, we round up, and the quotient obtained with a calculator is $(L_{p-m} - 1)$, since

$$F_p/F_m = L_{p-m} - F_{p-2m}/F_m = (L_{p-m} - 1) + (F_m - F_{p-2m})/F_m.$$

In the calculator case, the positive remainder is

$$F_m - F_{p-2m} = (F_{m-2} + F_{m-4}) + F_{m-6} + \ldots \pm F_{p-2m} = L_{m-3} + \ldots$$

Equation (15a) yields similar results.

Lucas Quotient Lemma 2. When a Lucas number L_p is divided by L_m, $3m > p > m > 0$, the quotient rounds off to a Lucas number. The (non-zero) remainder is either a Lucas number or its negative.

Proof: Apply Equation (17a) from [6] and analyze as in Lemma 1:

$$L_{n+m} + (-1)^m L_{n-m} = L_m L_n.$$

From [5], the Fibonacci and Lucas sequences are the only Fibonacci-like sequences possessing the property that division of a member of the sequence by a (non-zero) member of that same sequence yields least positive or negative residues that are either zero or a member of the original sequence.

For the Fibonacci-like sequence defined by $G_{n+1} = G_n + G_{n-1}$, G_0, G_1 arbitrary positive integers, neither the Lucas quotient property nor the remainder property holds in general. For example, for the sequence arising from $G_0 = 7$, $G_1 = 3$, $\{\ldots, 26, -15, 11, -4, 7, 3, 10, 13, 23, 36, 59, \ldots\}$, division of 59 by 10 gives 5 remainder 9 or 6 remainder (-1); 9 and (-1) do not appear in the sequence. While the sequences $\{G_n\}$ have the property that $\{G_n\}$ is congruent to a sequence made of the original sequence and negatives of those values, $G_n \equiv \pm G_r \pmod{G_k}$, those subsequences are actually remainders of the divisor when G_n/G_k for *only* the Fibonacci and Lucas sequences [5].

At first glance, Eq. (10a) from [6] seems to apply:

$$G_{n+m} + (-1)^m G_{n-m} = L_m G_n.$$

If m is odd and $G_{n-m} < G_n$, there are some cases of Lucas quotients paired with remainders within the sequence $\{G_n\}$. However, $\{G_n\}$ lacks the symmetry about G_0 of the Fibonacci sequence.

References

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